Coherence and polarization properties of far fields generated by quasi-homogeneous planar electromagnetic sources

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In studies of radiation from partially coherent sources the so-called quasi-homogeneous (QH) model sources have been very useful, for instance in elucidating the behavior of fields produced by thermal sources. The analysis of the fields generated by such sources has, however, been largely carried out in the framework of scalar wave theory. In this paper we generalize the concept of the QH source to the domain of the electromagnetic theory, and we derive expressions for the elements of the cross-spectral density matrix, for the spectral density, the spectral degree of coherence, the degree of polarization, and the Stokes parameters of the far field generated by planar QH sources of uniform states of polarization. We then derive reciprocity relations analogous to those familiar in connection with the QH scalar sources. We illustrate the results by determining the properties of the far field produced by transmission of an electromagnetic beam through a system of spatial light modulators. © 2005 Optical Society of America

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1. INTRODUCTION

The so-called quasi-homogeneous (QH) sources (see Ref. 1, Sec. 5.3.2) are important models for many partially coherent sources found in nature or developed in laboratories. Use of such models has clarified the behavior of Lambertian sources[2] as well as of secondary sources that are encountered in scattering theory.[3–7]

Up to now almost all investigations concerning QH sources and the fields that they generate have been based on scalar wave theory. Perhaps the most important and best-known result derived from the scalar QH model is the van Cittert–Zernike theorem and its generalization (see Refs. 1 and 8), which give a simple expression for the degree of coherence at a pair of points in the field propagated from such a source. When applied to points in the far zone the generalized van Cittert–Zernike theorem expresses a reciprocity between the intensity distribution across the source and the degree of coherence of the far field. There is a second less well-known reciprocity relation for fields generated by such sources that expresses the spectral density distribution in the far zone in terms of the spectral degree of coherence of the source (see Ref. 1, Sec. 5.3.2).

In this paper we generalize the concept of the quasi-homogeneous source to the domain of the electromagnetic theory with the help of the recently introduced 2×2 cross-spectral density matrix of the electric field.[9–11] We then show that a beam generated by an electromagnetic QH source that is uniformly polarized, i.e., that has the same state of polarization at each point, obeys two reciprocity relations analogous to those that are familiar for scalar QH sources. We also derive expressions for the spectral degree of polarization and the spectral Stokes parameters of the far field that are generated by such sources. We illustrate the results by plots that show the behavior of the far field produced by transmission of such a beam through a system involving spatial light modulators.

2. QUASI-HOMOGENEOUS ELECTROMAGNETIC SOURCE

Consider a fluctuating, planar, secondary electromagnetic source, located in the plane $z=0$ and radiating into the

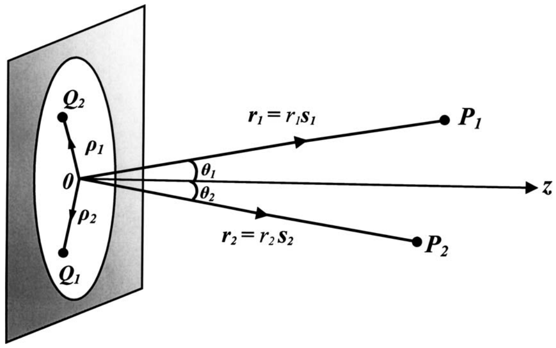


Fig. 1. Illustration of the notation.

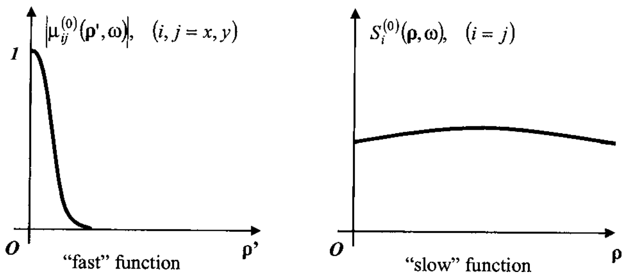


Fig. 2. Illustration of the concept of a QH source.

half-space $z>0$. We assume that the radiated field is beamlike, propagating close to the z axis, and that the source fluctuations are represented by a statistical ensemble that is stationary, at least in the wide sense. The second-order correlation properties of the source may be characterized by the 2×2 electric cross-spectral density matrix:[9]

$$\overleftrightarrow{W}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\begin{bmatrix} W_{xx}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) & W_{xy}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) \\ W_{yx}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) & W_{yy}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)\end{bmatrix}. \tag{2.1}$$

In this formula $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are the two-dimensional position vectors of two points Q_1 and Q_2 in the source plane (see Fig. 1), ω denotes the frequency, and the elements of the W-matrix are

$$W_{ij}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\langle E_i^*(\boldsymbol{\rho}_1,\omega)E_j(\boldsymbol{\rho}_2,\omega)\rangle \quad (i=x,y;\quad j=x,y). \tag{2.2}$$

Here E_i and E_j denote the Cartesian components of a typical member of the statistical ensemble of the electric field in two mutually orthogonal x and y directions perpendicular to the z-axis (the axis of the beam), and the angle brackets denote the average, taken over an ensemble of realizations of the electric field, in the sense of coherence theory in the space-frequency domain (Ref. 1, Sec. 4.7.1).

The electric cross-spectral density matrix representing an electromagnetic QH source may be obtained as a straightforward generalization of the cross-spectral density function of a scalar QH source:[1,8]

$$W_{ij}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\sqrt{S_i^{(0)}((\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)/2,\omega)}\sqrt{S_j^{(0)}((\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)/2,\omega)} \times\mu_{ij}^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega), \quad (i=x,y;\quad j=x,y), \tag{2.3}$$

where $S_i^{(0)}(\boldsymbol{\rho},\omega)=W_{ii}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)$ and $S_j^{(0)}(\boldsymbol{\rho},\omega)=W_{jj}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)$ are the spectral densities of the components E_i and E_j and $\mu_{ij}^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega)$ represent the correlations between these components.[12] It is assumed that $S_i^{(0)}(\boldsymbol{\rho},\omega)$ $(i=x,y)$ varies much more slowly with $\boldsymbol{\rho}$ than $\mu_{ij}^{(0)}(\boldsymbol{\rho}',\omega)$ varies with $\boldsymbol{\rho}'$ (see Fig. 2).

The spectral density $S^{(0)}(\boldsymbol{\rho},\omega)$, the spectral degree of coherence $\eta^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)$, and the spectral degree of polarization $P^{(0)}(\boldsymbol{\rho},\omega)$ of an electromagnetic beam are given by the formulas[9]

$$S^{(0)}(\boldsymbol{\rho},\omega) = \mathrm{Tr}\,\overleftrightarrow{W}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega), \tag{2.4}$$

$$\eta^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \frac{\mathrm{Tr}\,\overleftrightarrow{W}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)}{\sqrt{S^{(0)}(\boldsymbol{\rho}_1,\omega)}\sqrt{S^{(0)}(\boldsymbol{\rho}_2,\omega)}}, \tag{2.5}$$

$$P^{(0)}(\boldsymbol{\rho},\omega) = \sqrt{1-\frac{4\,\mathrm{Det}\,\overleftrightarrow{W}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)}{[\mathrm{Tr}\,\overleftrightarrow{W}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)]^2}}, \tag{2.6}$$

where Tr denotes the trace and Det the determinant. For a quasi-homogeneous electromagnetic source whose cross-spectral density matrix $\overleftrightarrow{W}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)$ is given by Eq. (2.3) expressions (2.4)–(2.6) become

$$S^{(0)}(\boldsymbol{\rho},\omega)=S_x^{(0)}(\boldsymbol{\rho},\omega)+S_y^{(0)}(\boldsymbol{\rho},\omega), \tag{2.7}$$

$$\eta^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\frac{S_x^{(0)}((\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)/2,\omega)\mu_{xx}^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega)+S_y^{(0)}((\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)/2,\omega)\mu_{yy}^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega)}{S^{(0)}((\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)/2,\omega)}, \tag{2.8}$$

$$P^{(0)}(\boldsymbol{\rho},\omega)=\frac{\sqrt{[S_x^{(0)}(\boldsymbol{\rho},\omega)-S_y^{(0)}(\boldsymbol{\rho},\omega)]^2+4S_x^{(0)}(\boldsymbol{\rho},\omega)S_y^{(0)}(\boldsymbol{\rho},\omega)|\mu_{xy}^{(0)}(0,\omega)|^2}}{S^{(0)}(\boldsymbol{\rho},\omega)}. \tag{2.9}$$

Formula (2.7) holds generally; for a QH source, $S^{(0)}(\boldsymbol{\rho},\omega)$ is a slow function of $\boldsymbol{\rho}$. The Stokes parameters, which are useful for characterizing the state of polarization of an electromagnetic field, may be expressed in terms of the elements of the cross-spectral density matrix by the formulas (cf. Ref. 13, Sec. 10.9.3)

$$\mathsf{S}_0^{(0)}(\boldsymbol{\rho},\omega) = W_{xx}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega) + W_{yy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega), \quad (2.10a)$$

$$\mathsf{S}_1^{(0)}(\boldsymbol{\rho},\omega) = W_{xx}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega) - W_{yy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega), \quad (2.10b)$$

$$\mathsf{S}_2^{(0)}(\boldsymbol{\rho},\omega) = 2\,\mathrm{Re}[W_{xy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)], \quad (2.10c)$$

$$\mathsf{S}_3^{(0)}(\boldsymbol{\rho},\omega) = 2\,\mathrm{Im}[W_{xy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)], \quad (2.10d)$$

Re and Im denoting the real and the imaginary parts, respectively. It will be useful to normalize the Stokes parameters by setting[14]

$$\mathsf{s}_i^{(0)}(\boldsymbol{\rho},\omega) = \mathsf{S}_i^{(0)}(\boldsymbol{\rho},\omega)/\mathsf{S}_0^{(0)}(\boldsymbol{\rho},\omega) \quad (i=1,2,3). \quad (2.11)$$

The normalized Stokes parameters $\mathsf{s}_i^{(0)}(\boldsymbol{\rho},\omega)$ $(i=1, 2, 3)$ associated with the QH source characterized by Eqs. (2.3) are readily shown to be given by the formulas

$$\mathsf{s}_1^{(0)}(\boldsymbol{\rho},\omega) = \frac{S_x^{(0)}(\boldsymbol{\rho},\omega) - S_y^{(0)}(\boldsymbol{\rho},\omega)}{S_x^{(0)}(\boldsymbol{\rho},\omega) + S_y^{(0)}(\boldsymbol{\rho},\omega)}, \quad (2.12a)$$

$$\mathsf{s}_2^{(0)}(\boldsymbol{\rho},\omega) = \frac{2\sqrt{S_x^{(0)}(\boldsymbol{\rho},\omega)}\sqrt{S_y^{(0)}(\boldsymbol{\rho},\omega)}\mathrm{Re}[\mu_{xy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)]}{S_x^{(0)}(\boldsymbol{\rho},\omega) + S_y^{(0)}(\boldsymbol{\rho},\omega)}, \quad (2.12b)$$

$$\mathsf{s}_3^{(0)}(\boldsymbol{\rho},\omega) = \frac{2\sqrt{S_x^{(0)}(\boldsymbol{\rho},\omega)}\sqrt{S_y^{(0)}(\boldsymbol{\rho},\omega)}\mathrm{Im}[\mu_{xy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)]}{S_x^{(0)}(\boldsymbol{\rho},\omega) + S_y^{(0)}(\boldsymbol{\rho},\omega)}. \quad (2.12c)$$

We will consider only sources for which the normalized spectral Stokes parameters are independent of $\boldsymbol{\rho}$. The spectral degree of polarization and the spectral polarization ellipse associated with the polarized portion of the beam will then be the same at every point of the source. We will call such sources *uniformly polarized*. In Appendix A we show that the source is uniformly polarized at frequency ω if and only if the spectral densities $S_x^{(0)}(\boldsymbol{\rho},\omega)$ and $S_y^{(0)}(\boldsymbol{\rho},\omega)$ are proportional to each other, i.e., if and only if[15]

$$S_y^{(0)}(\boldsymbol{\rho},\omega) = \alpha(\omega) S_x^{(0)}(\boldsymbol{\rho},\omega) \quad (2.13a)$$

and if, in addition, the correlation coefficient $\mu_{xy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)$ is independent of position, i.e.,

$$\mu_{xy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega) \equiv \mu_{xy}^{(0)}(\omega). \quad (2.13b)$$

It follows from Eqs. (2.4), (2.13a), and (2.13b) that

$$S_x^{(0)}(\boldsymbol{\rho},\omega) = \frac{1}{1+\alpha(\omega)} S^{(0)}(\boldsymbol{\rho},\omega),$$

$$S_y^{(0)}(\boldsymbol{\rho},\omega) = \frac{\alpha(\omega)}{1+\alpha(\omega)} S^{(0)}(\boldsymbol{\rho},\omega). \quad (2.14)$$

On substituting from Eqs. (2.14) into Eq. (2.3) we find that the elements of the cross-spectral density matrix of the source that satisfy conditions (2.13a) and (2.13b) are given by

$$W_{ij}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \alpha_{ij}(\omega) S^{(0)}((\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)/2,\omega)\mu_{ij}^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega), \quad (2.15)$$

where the coefficients $\alpha_{ij}(\omega)$ are

$$\alpha_{ij}(\omega) = \begin{cases} \dfrac{1}{1+\alpha(\omega)} & \text{when } i=j=x \\ \dfrac{\sqrt{\alpha(\omega)}}{1+\alpha(\omega)} & \text{when } i \neq j \\ \dfrac{\alpha(\omega)}{1+\alpha(\omega)} & \text{when } i=j=y \end{cases}. \quad (2.16)$$

The spectral degree of coherence of the source is then given by the expression

$$\eta^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \alpha_{xx}(\omega)\mu_{xx}^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega) + \alpha_{yy}(\omega)\mu_{yy}^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega). \quad (2.17)$$

Expressions for the normalized Stokes parameters of the source are derived in Appendix A.

3. THE SPECTRAL DENSITY AND THE SPECTRAL DEGREE OF COHERENCE OF THE FAR FIELD. TWO RECIPROCITY RELATIONS

It can be shown by a straightforward analogy of the argument used in deriving Eqs. (5.3-4) and (5.3-5) of Ref. 1 that the elements of the cross-spectral density matrix of the electric field in the far-zone are given by the expressions (see Fig. 1)

$$W_{ij}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega) = (2\pi k)^2 \cos\theta_1 \cos\theta_2 \times \tilde{W}_{ij}^{(0)}(-k\mathbf{s}_{1\perp},k\mathbf{s}_{2\perp},\omega)\frac{\exp[ik(r_2-r_1)]}{r_1 r_2}, \quad (3.1)$$

where

$$\tilde{W}_{ij}^{(0)}(\mathbf{f}_1,\mathbf{f}_2,\omega) = \frac{1}{(2\pi)^4}\int\int W_{ij}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)\exp[-i(\mathbf{f}_1\cdot\boldsymbol{\rho}_1 + \mathbf{f}_2\cdot\boldsymbol{\rho}_2)]\mathrm{d}^2\rho_1\mathrm{d}^2\rho_2 \quad (3.2)$$

is the four-dimensional spatial Fourier transform of $W_{ij}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)$. In Eq. (3.1), $\mathbf{s}_1$ and $\mathbf{s}_2$ are unit vectors along $\mathbf{r}_1$ and $\mathbf{r}_2$ and $\mathbf{s}_{1\perp}$ and $\mathbf{s}_{2\perp}$ are the projections, considered as four-dimensional vectors, of the unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ onto the source plane. Each integration in Eq. (3.2) extends over the source.

On substituting from Eq. (2.15) into Eq. (3.2) we obtain for the Fourier transforms of the elements of the cross-spectral density matrix of a uniformly polarized QH source the expressions

$$\begin{aligned}\tilde{W}_{ij}^{(0)}(\mathbf{f}_1,\mathbf{f}_2,\omega) &= \frac{\alpha_{ij}(\omega)}{(2\pi)^4}\int\int S^{(0)}((\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)/2,\omega) \\ &\quad\times\mu_{ij}^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega)\exp[-i(\mathbf{f}_1\cdot\boldsymbol{\rho}_1 + \mathbf{f}_2\cdot\boldsymbol{\rho}_2)]\mathrm{d}^2\rho_1\mathrm{d}^2\rho_2 \\ &= \alpha_{ij}(\omega)\tilde{S}^{(0)}(\mathbf{f}_1+\mathbf{f}_2,\omega)\tilde{\mu}_{ij}^{(0)}((\mathbf{f}_2-\mathbf{f}_1)/2,\omega)\end{aligned}$$

$$(i=x,y;\quad j=x,y), \quad (3.3)$$

where $\alpha_{ij}(\omega)$ are the coefficients defined in Eq. (2.16) and

$$\tilde{S}^{(0)}(\mathbf{f},\omega) = \frac{1}{(2\pi)^2}\int S^{(0)}(\boldsymbol{\rho},\omega)\exp[-i\mathbf{f}\cdot\boldsymbol{\rho}]\mathrm{d}^2\rho, \quad (3.4)$$

$$\tilde{\mu}_{ij}^{(0)}(\mathbf{f},\omega) = \frac{1}{(2\pi)^2}\int \mu_{ij}^{(0)}(\boldsymbol{\rho},\omega)\exp(-i\mathbf{f}\cdot\boldsymbol{\rho})\mathrm{d}^2\rho. \quad (3.5)$$

Formula (3.3) is derived in Appendix B.

On substituting from Eq. (3.3) into Eq. (3.1) we obtain for the elements of the cross-spectral density matrix in the far zone the expressions

$$W_{ij}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega) = (2\pi k)^2\cos\theta_1\cos\theta_2\alpha_{ij}(\omega) \times \tilde{S}^{(0)}(k(\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp}),\omega) \times \tilde{\mu}_{ij}^{(0)}(k(\mathbf{s}_{2\perp}+\mathbf{s}_{1\perp})/2,\omega)\frac{\exp[ik(r_2-r_1)]}{r_1 r_2}. \quad (3.6)$$

The spectral density $S^{(\infty)}(r\mathbf{s},\omega)$ of the far field is given by the expression

$$S^{(\infty)}(r\mathbf{s},\omega) \equiv \mathrm{Tr}\,\overleftrightarrow{W}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega) = (2\pi k)^2\cos^2\theta\tilde{S}^{(0)}(0,\omega) \times[\alpha_{xx}(\omega)\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_{\perp},\omega)+\alpha_{yy}(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_{\perp},\omega)]\frac{1}{r^2}, \quad (3.7)$$

where θ is the angle that the unit vector $\mathbf{s}$ makes with the positive z axis. We also have, on taking the Fourier transform of Eq. (2.17), that

$$\tilde{\eta}^{(0)}(k\mathbf{s}_{\perp},\omega) = \alpha_{xx}(\omega)\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_{\perp},\omega)+\alpha_{yy}(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_{\perp},\omega). \quad (3.8)$$

On substituting from Eq. (3.8) into Eq. (3.7) we obtain for the spectral density of the electric field in the far zone the expression

$$S^{(\infty)}(r\mathbf{s},\omega) = (2\pi k)^2\cos^2\theta\tilde{S}^{(0)}(0,\omega)\tilde{\eta}^{(0)}(k\mathbf{s}_{\perp},\omega)\frac{1}{r^2}. \quad (3.9)$$

Equation (3.9) expresses a reciprocity relation that holds for beams generated by a planar, secondary, uniformly polarized electromagnetic QH source, namely, *the spectral density of the far field is proportional to the product of the Fourier transform of the spectral degree of coherence of the field across the source and of* $\cos^2\theta/r^2$.

Next we consider the spectral degree of coherence of the far field at points $\mathbf{r}_1=r_1\mathbf{s}_1$ and $\mathbf{r}_2=r_2\mathbf{s}_2$, viz.,

$$\eta^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_1,\omega) = \frac{\mathrm{Tr}\,\overleftrightarrow{W}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega)}{\sqrt{\mathrm{Tr}\,\overleftrightarrow{W}^{(\infty)}(r_1\mathbf{s}_1,r_1\mathbf{s}_1,\omega)}\sqrt{\mathrm{Tr}\,\overleftrightarrow{W}^{(\infty)}(r_2\mathbf{s}_2,r_2\mathbf{s}_2,\omega)}}. \quad (3.10)$$

With the help of Eq. (3.6) one finds that

$$\eta^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega) = \frac{\alpha_{xx}(\omega)\tilde{\mu}_{xx}^{(0)}(k(\mathbf{s}_{2\perp}+\mathbf{s}_{1\perp})/2,\omega)+\alpha_{yy}(\omega)\tilde{\mu}_{yy}^{(0)}(k(\mathbf{s}_{2\perp}+\mathbf{s}_{1\perp})/2,\omega)}{\sqrt{\alpha_{xx}(\omega)\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_{1\perp},\omega)+\alpha_{yy}(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_{1\perp},\omega)}\sqrt{\alpha_{xx}(\omega)\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_{2\perp},\omega)+\alpha_{yy}(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_{2\perp},\omega)}} \times\frac{\tilde{S}^{(0)}(k(\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp}),\omega)}{\tilde{S}^{(0)}(0,\omega)}\exp[ik(r_2-r_1)]. \quad (3.11)$$

The assumption that the correlation coefficients $\mu_{ii}^{(0)}(\boldsymbol{\rho}',\omega)$ $(i=x,y)$ are "fast" functions of $\boldsymbol{\rho}'$ across the source implies that their Fourier transforms $\tilde{\mu}_{ii}^{(0)}(\mathbf{f}',\omega)$ are "slow" functions of $\mathbf{f}'$. We can, therefore, make the approximations

$$\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_{1\perp},\omega) \approx \tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_{2\perp},\omega)\approx\tilde{\mu}_{xx}^{(0)}(k(\mathbf{s}_{2\perp}+\mathbf{s}_{1\perp})/2,\omega), \quad (3.12a)$$

$$\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_{1\perp},\omega) \approx \tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_{2\perp},\omega)\approx\tilde{\mu}_{yy}^{(0)}(k(\mathbf{s}_{2\perp}+\mathbf{s}_{1\perp})/2,\omega), \quad (3.12b)$$

and it follows that the first factor in Eq. (3.11) can be approximated by unity. We then obtain for the complex degree of coherence $\eta^{(\infty)}$ of the far field the formula

$$\eta^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega) = \frac{\tilde{S}^{(0)}(k(\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp}),\omega)}{\tilde{S}^{(0)}(0,\omega)}\exp[ik(r_2-r_1)]. \quad (3.13)$$

This formula expresses another reciprocity relation pertaining to radiation from uniformly polarized planar QH electromagnetic sources, namely, *the spectral degree of coherence* $\eta^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega)$ *of the far field generated by an electromagnetic, uniformly polarized, QH source is, apart from the phase factor* $k(r_2-r_1)$*, proportional to the four-dimensional spatial Fourier transform of the spectral density of the electric field in the source plane.*[16]

We can summarize this part of the analysis by saying that we have shown that the two reciprocity relations associated with beams produced by scalar planar QH sources (Ref. 1, Sec. 5.3.2) have analogs for beams generated by planar uniformly polarized QH electromagnetic sources.

4. POLARIZATION PROPERTIES OF THE FAR FIELD

The full description of the polarization properties of a random electromagnetic beam at some point $\mathbf{r}$ requires the specification of the beam's degree of polarization and of the state of polarization of its polarized portion.

The spectral degree of polarization of a random electromagnetic beam at a point $\mathbf{r}=r\mathbf{s}$ in the far zone is given by the expression

$$P^{(\infty)}(r\mathbf{s},\omega)=\sqrt{1-\frac{4\,\mathrm{Det}\,\overleftrightarrow{W}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)}{[\mathrm{Tr}\,\overleftrightarrow{W}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)]^2}}, \quad (4.1a)$$

or, more explicitly, by the expression

$$P^{(\infty)}(r\mathbf{s},\omega)=\frac{\sqrt{[W_{xx}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)-W_{yy}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)]^2+4|W_{xy}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)|^2}}{W_{xx}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)+W_{yy}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)}. \quad (4.1b)$$

On substituting into Eq. (4.1b) from Eq. (3.6) for the elements of the matrix $\overleftrightarrow{W}^{(\infty)}$ of the far field generated by a uniformly polarized electromagnetic QH source evaluated at a point $r_1\mathbf{s}_1=r_2\mathbf{s}_2\equiv r\mathbf{s}$, one finds that the spectral degree of polarization of the far field is given by the formula

$$P^{(\infty)}(r\mathbf{s},\omega)=\frac{\sqrt{[\alpha_{xx}(\omega)\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_\perp,\omega)-\alpha_{yy}(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_\perp,\omega)]^2+4[\alpha_{xy}(\omega)\tilde{\mu}_{xy}^{(0)}(k\mathbf{s}_\perp,\omega)]^2}}{\alpha_{xx}(\omega)\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_\perp,\omega)+\alpha_{yy}(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_\perp,\omega)}. \quad (4.2)$$

Using definition (2.16), formula (4.2) may be rewritten in the form

$$P^{(\infty)}(r\mathbf{s},\omega)=\frac{\sqrt{[\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_\perp,\omega)-\alpha(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_\perp,\omega)]^2+4\alpha(\omega)[\tilde{\mu}_{xy}^{(0)}(k\mathbf{s}_\perp,\omega)]^2}}{\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_\perp,\omega)+\alpha(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_\perp,\omega)}. \quad (4.3)$$

This formula expresses the spectral degree of polarization of the far field generated by a uniformly polarized planar QH electromagnetic source in terms of the four-dimensional spatial Fourier transforms of the correlation coefficients $\mu_{ij}^{(0)}(\boldsymbol{\rho}',\omega)$ of the electric field in the source plane and the factor $\alpha(\omega)$, defined by Eq. (2.13a).

The normalized Stokes parameters of the far field $\mathsf{s}_i^{(\infty)}$ $(i=1,2,3)$ are related to the elements of the cross-spectral density matrix $\overleftrightarrow{W}^{(\infty)}$ in the same way as in the source plane. For uniformly polarized QH sources, application of Eqs. (2.16) and (3.6) leads to the far-field normalized Stokes parameters as[18]

$$\mathsf{s}_1^{(\infty)}(r\mathbf{s},\omega)=\frac{\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_\perp,\omega)-\alpha(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_\perp,\omega)}{\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_\perp,\omega)+\alpha(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_\perp,\omega)}, \quad (4.4a)$$

$$\mathsf{s}_2^{(\infty)}(r\mathbf{s},\omega)=\frac{2\sqrt{\alpha(\omega)}\mathrm{Re}[\tilde{\mu}_{xy}^{(0)}(k\mathbf{s}_\perp,\omega)]}{\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_\perp,\omega)+\alpha(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_\perp,\omega)}, \quad (4.4b)$$

$$\mathsf{s}_3^{(\infty)}(r\mathbf{s},\omega)=\frac{2\sqrt{\alpha(\omega)}\mathrm{Im}[\tilde{\mu}_{xy}^{(0)}(k\mathbf{s}_\perp,\omega)]}{\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_\perp,\omega)+\alpha(\omega)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_\perp,\omega)}. \quad (4.4c)$$

To summarize, formulas (4.3) and (4.4) express the spectral degree of polarization and the normalized spectral Stokes parameters of the far field generated by a planar uniformly polarized QH source in terms of the Fourier transforms $\tilde{\mu}_{xx}^{(0)}$, $\tilde{\mu}_{yy}^{(0)}$, and $\tilde{\mu}_{xy}^{(0)}$ of the correlation coefficients of the field components in the source plane and in terms of the ratio $\alpha(\omega)$ of the spectral densities $S_x^{(0)}(\boldsymbol{\rho},\omega)$ and $S_y^{(0)}(\boldsymbol{\rho},\omega)$ of the components of the electric vector in that plane [Eq. (2.13a)].

5. EXAMPLE: THE FAR FIELD GENERATED BY A MODEL SOURCE

We will illustrate our main results by determining the spectral density, the spectral degree of coherence, the spectral degree of polarization, and the normalized Stokes parameters of the far field produced by sources generated by a technique described in Ref. 19 (see also Ref. 20). According to that technique, the so-called electromagnetic Gaussian Schell-model source can be synthesized from two coherent, linearly polarized plane waves transmitted through two mutually correlated phase-only liquid-crystal spatial light modulators (SLMs), placed in the arms of a Mach–Zehnder interferometer.

It was shown in Ref. 19 that the elements of the cross-spectral density matrix of the field generated by such a device has the form of our Eq. (2.15), with the spectral density $S^{(0)}$ and the correlation coefficients $\mu_{ij}^{(0)}$ being Gaussian functions, viz.,

$$S^{(0)}(\boldsymbol{\rho},\omega) = \frac{A(\omega)}{2}\exp\left[-\frac{\rho^2}{2\sigma^2(\omega)}\right] \quad (j = x,y), \quad (5.1)$$

$$\mu_{ij}^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega) = B_{ij}(\omega)\exp\left[-\frac{|\boldsymbol{\rho}_2-\boldsymbol{\rho}_1|^2}{2\delta_{ij}^2(\omega)}\right] \quad (i = x,y;\ j=x,y). \quad (5.2)$$

Here the coefficients $B_{ij}(\omega)$ are, in general, complex numbers. Since in this example the variances $\sigma_x(\omega)$ and $\sigma_y(\omega)$ are the same, we denoted each of them by $\sigma(\omega)$. This condition ensures that the source is uniformly polarized.[21]

There are various constraints on the parameters entering expressions (5.1) and (5.2). First, the general properties (e.g., nonnegative definiteness) of the cross-spectral density matrix of the field in the source plane impose certain conditions on the parameters of the source. The conditions have been derived in Ref. 22 for electromagnetic Gaussian Schell-model sources. In addition, in order to ensure that the field produced by such a source is beamlike, the following conditions must be satisfied (see Ref. 23):

$$\frac{1}{4\sigma^2(\omega)} + \frac{1}{\delta_{xx}^2(\omega)} \ll \frac{c^2}{2\omega^2} = \frac{2\pi^2}{\lambda^2},$$

$$\frac{1}{4\sigma^2(\omega)} + \frac{1}{\delta_{yy}^2(\omega)} \ll \frac{c^2}{2\omega^2} = \frac{2\pi^2}{\lambda^2}. \tag{5.3}$$

From now on we will omit the explicit dependence of all the parameters on the frequency. In order that the electromagnetic Gaussian Schell-model source generated by means of the system of the SLMs will be quasi-homogeneous,

$$\sigma \gg \delta_{ij} \quad (i = x,y, \quad j = x,y). \tag{5.4}$$

To determine the behavior of the far field, we evaluate first the four-dimensional Fourier transform of the spectral density (5.1) and of the correlation coefficients (5.2). One readily finds that

$$\tilde{S}^{(0)}(\mathbf{f},\omega) = \frac{A\sigma^2}{4\pi}\exp\left[-\frac{\sigma^2}{2}f^2\right], \tag{5.5}$$

$$\tilde{\mu}_{ij}^{(0)}(\mathbf{f}',\omega) = \frac{\delta_{ij}^2}{2\pi}B_{ij}\exp\left[-\frac{\delta_{ij}^2}{2}f'^2\right]. \tag{5.6}$$

On substituting from Eqs. (5.5) and (5.6) into Eq. (3.7) we obtain the following expression for the spectral density of far field:

$$S^{(\infty)}(r\mathbf{s},\omega) = \frac{Ak^2\sigma^2}{2r^2[1+\alpha]}[\delta_{xx}^2\exp(-\delta_{xx}^2k^2\theta^2/2) + \alpha\delta_{yy}^2\exp(-\delta_{yy}^2k^2\theta^2/2)]. \tag{5.7}$$

As before, θ, the angle that the vector $\mathbf{r}=r\mathbf{s}$ makes with the z axis, is assumed to be small, so that $\cos\theta\approx 1$, $|\mathbf{s}_\perp|^2 \equiv \sin^2\theta \approx \theta^2$.[24]

On substituting from Eq. (5.5) into formula (3.13) we obtain for the spectral degree of coherence of the far field the expression

$$\eta^{(\infty)}(r_1\mathbf{s}_1, r_2\mathbf{s}_2,\omega) = \exp[-\sigma^2k^2|\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp}|^2/2]\exp[ik(r_2-r_1)]. \tag{5.8}$$

From Eqs. (5.6) and (4.3) we readily obtain the following expression for the spectral degree of polarization of the far field:

$$P^{(\infty)}(r\mathbf{s},\omega) = \frac{\sqrt{[\delta_{xx}^2\exp(-\delta_{xx}^2k^2\theta^2/2) - \alpha\delta_{yy}^2\exp(-\delta_{yy}^2k^2\theta^2/2)]^2 + 4\alpha\delta_{xy}^4|B_{xy}|^2\exp(-\delta_{xy}^2k^2\theta^2)}}{\delta_{xx}^2\exp(-\delta_{xx}^2k^2\theta^2/2) + \alpha\delta_{yy}^2\exp(-\delta_{yy}^2k^2\theta^2/2)}. \tag{5.9}$$

Finally, it follows from Eqs. (4.4) and (5.6) that the normalized spectral Stokes parameters of the far field generated by a planar electromagnetic Gaussian Schell-model source are given by the expressions

$$\mathsf{s}_1^{(\infty)}(r\mathbf{s},\omega) = \frac{\delta_{xx}^2\exp(-\delta_{xx}^2k^2\theta^2/2) - \alpha\delta_{yy}^2\exp(-\delta_{yy}^2k^2\theta^2/2)}{\delta_{xx}^2\exp(-\delta_{xx}^2k^2\theta^2/2) + \alpha\delta_{yy}^2\exp(-\delta_{yy}^2k^2\theta^2/2)}, \tag{5.10a}$$

$$\mathsf{s}_2^{(\infty)}(r\mathbf{s},\omega) = \frac{2\sqrt{\alpha}\delta_{xy}^2|B_{xy}|\cos\varphi_{xy}\exp(-\delta_{xy}^2k^2\theta^2/2)}{\delta_{xx}^2\exp(-\delta_{xx}^2k^2\theta^2/2) + \alpha\delta_{yy}^2\exp(-\delta_{yy}^2k^2\theta^2/2)}, \tag{5.10b}$$

$$\mathsf{s}_3^{(\infty)}(r\mathbf{s},\omega) = \frac{2\sqrt{\alpha}\delta_{xy}^2|B_{xy}|\sin\varphi_{xy}\exp(-\delta_{xy}^2k^2\theta^2/2)}{\delta_{xx}^2\exp(-\delta_{xx}^2k^2\theta^2/2) + \alpha\delta_{yy}^2\exp(-\delta_{yy}^2k^2\theta^2/2)}, \tag{5.10c}$$

where φ_{xy} is the phase of the coefficient B_{xy} [see Eq. (5.2)].

Figures 3–5 show the behavior of the spectral density, normalized by its axial value, of the spectral degree of polarization and of the normalized Stokes parameters of the electric field in the far zone, generated by a uniformly polarized QH electromagnetic source, specified by Eqs. (5.1) and (5.2).

In Figs. 3 and 4 the off-diagonal elements of the cross-spectral density matrix are both zero, i.e., $B_{xy}=B_{yx}=0$. We see that although the source is unpolarized, the far field is partially polarized and the degree of polarization varies with the angle θ and also with the difference $\delta_{xx}-\delta_{yy}$ between the rms widths of the source correlation coefficients [see Eq. (5.2)]. One can show from Eq. (5.9) that when $B_{xy}=B_{yx}=0$, then $P^{(\infty)}(r\mathbf{s},\omega)=|\mathsf{s}_1^{(\infty)}(r\mathbf{s},\omega)|$, while $\mathsf{s}_2^{(\infty)}(r\mathbf{s},\omega) = \mathsf{s}_3^{(\infty)}(r\mathbf{s},\omega)\equiv 0$. Figure 3 pertains to the case for which changes of the degree of polarization in the far zone occur only near the beam edge, while Fig. 4 illustrates the case for which changes in the degree of polarization occur within the beam cross section.

Figure 5 illustrates the far-zone properties of a field generated by a planar, uniformly polarized, quasi-homogeneous Gaussian Schell-model source, which is characterized by a cross-spectral density matrix $\overleftrightarrow{W}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)$ whose off-diagonal elements are not necessarily zero. Unlike sources with "diagonal" cross-spectral density matrices (Figs. 3 and 4), the field generated by this source cannot be treated as a superposition of two orthogonally polarized scalar fields. It is evident from Eqs. (5.9) and (5.10) that the state of polarization of the far field generated by a QH source represented by a cross-spectral density matrix with nonvanishing offdiagonal elements has nontrivial dependence on angle θ.

The spectral density and the polarization properties of the far field generated by a uniformly polarized QH source, characterized by Eq. (5.1)–(5.4), possess rotational symmetry. Figure 6 illustrates this property for the case represented in Fig. 5. The black-to-white (gray-scale) background in this diagram corresponds to the values of the spectral density of the far field (normalized by its axial value) in the range from 0 to 1. The black ellipses superimposed on this background represent the spectral polarization ellipses of the fully polarized portion of the far field. The parameters of each of the spectral polarization ellipses can be readily calculated from the Stokes parameters [see Eq. (5.10)] with the help of well-known relations (see Ref. 13, Sec. 10.9.3, see also Ref. 25).

APPENDIX A: NECESSARY AND SUFFICIENT CONDITIONS FOR A UNIFORMLY POLARIZED PLANAR QH SOURCE

We will show that a QH source is uniformly polarized at frequency ω if and only if the spectral densities $S_x^{(0)}(\boldsymbol{\rho},\omega)$ and $S_y^{(0)}(\boldsymbol{\rho},\omega)$ are proportional to each other, i.e., if

$$S_y^{(0)}(\boldsymbol{\rho},\omega) = \alpha(\omega) S_x^{(0)}(\boldsymbol{\rho},\omega) \tag{A1a}$$

and if, in addition, the correlation coefficient $\mu_{xy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)$ is independent of position, i.e., if

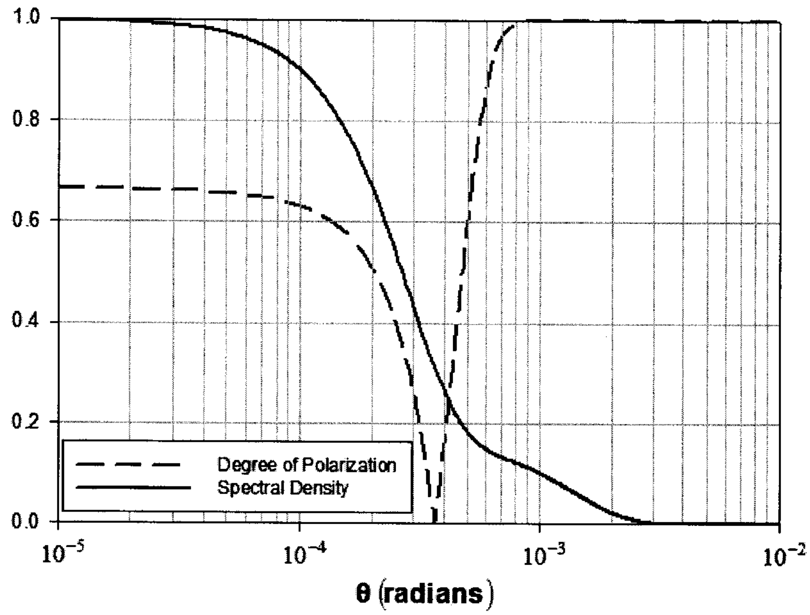


Fig. 3. The spectral density $S(r\mathbf{s},\omega)$ and the spectral degree of polarization of the far field generated by transmission of a linearly polarized beam through a system of spatial light modulators that produces a uniformly polarized QH field with parameters $\alpha=1$, $\sigma=1$ mm, $B_{xy}=0$, $\delta_{xx}=0.1$ mm, $\delta_{yy}=0.5$ mm, $\lambda=0.6328$ μm. The spectral density is normalized by its on-axis value.

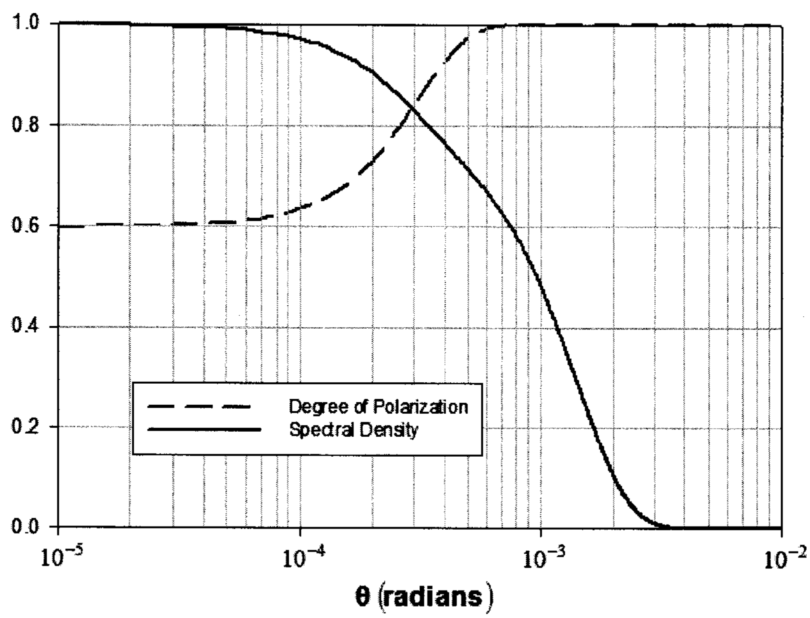
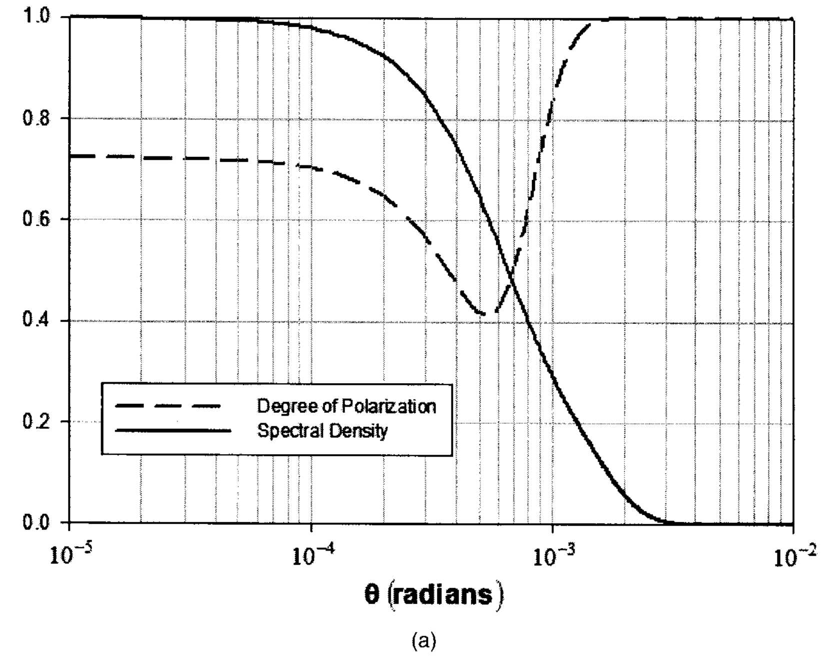


Fig. 4. The curves represent the same quantities as in Fig. 3, but the value of parameter α is 0.05.



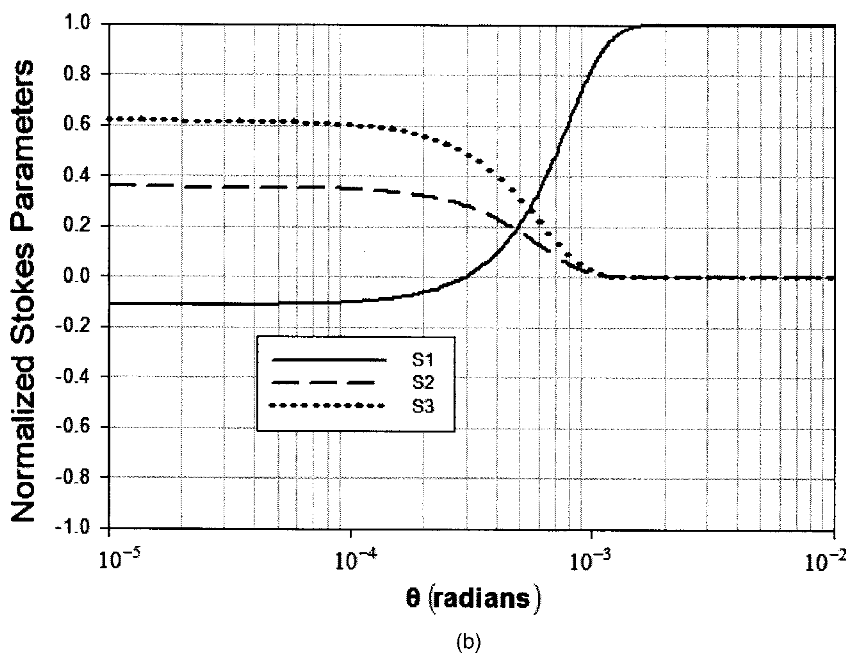


Fig. 5. (a) The degree of ploarization $P(r\mathbf{s},\omega)$ and the spectral density $S(r\mathbf{s},\omega)$ of the far field generated by the source with parameters $\alpha=0.44$, $\sigma=1$ mm, $B_{xy}=0.525$, $\delta_{xx}=0.15$ mm, $\delta_{yy}=0.25$ mm, $\delta_{xy}=0.3$ mm, $\lambda=0.6328$ μm, $\varphi_{xy}=\lambda/6$. (b) Normalized Stokes parameters of the far field generated by the same source as in (a).

$$\mu_{xy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega) \equiv \mu_{xy}^{(0)}(\omega). \tag{A1b}$$

First, if conditions (A1a) and (A1b) hold, then one finds, with the help of the Eqs. (2.10), that the normalized Stokes parameters, given by the expressions

$$\mathrm{s}_1^{(0)}(\omega) = \frac{1-\alpha(\omega)}{1+\alpha(\omega)}, \tag{A2a}$$

$$\mathrm{s}_2^{(0)}(\omega) = \frac{2\sqrt{\alpha(\omega)}\mathrm{Re}[\mu_{xy}^{(0)}(\omega)]}{1+\alpha(\omega)}, \tag{A2b}$$

$$\mathrm{s}_3^{(0)}(\omega) = \frac{2\sqrt{\alpha(\omega)}\mathrm{Im}[\mu_{xy}^{(0)}(\omega)]}{1+\alpha(\omega)}, \tag{A2c}$$

are independent of position. Hence, such a source is uniformly polarized.

Conversely, suppose that the state of polarization is uniform across the source. The normalized Stokes parameters, given by the formulas (2.12), are then independent of $\boldsymbol{\rho}$, say,

$$\mathrm{s}_i^{(0)}(\boldsymbol{\rho},\omega) \equiv \mathrm{s}_i^{(0)}(\omega) \quad (i=1,2,3) \tag{A3}$$

Equations Eqs. (2.12) and (A3) then imply that

$$\frac{S_x^{(0)}(\boldsymbol{\rho},\omega)}{S_y^{(0)}(\boldsymbol{\rho},\omega)}=\frac{1+\mathsf{s}_1^{(0)}(\omega)}{1-\mathsf{s}_1^{(0)}(\omega)}\equiv\alpha(\omega), \quad \text{(A4)}$$

and that

$$\mathrm{Re}[\mu_{xy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)]=\frac{\mathsf{s}_2^{(0)}(\omega)}{\sqrt{1-[\mathsf{s}_1^{(0)}(\omega)]^2}}, \quad \text{(A5a)}$$

$$\mathrm{Im}[\mu_{xy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)]=\frac{\mathsf{s}_3^{(0)}(\omega)}{\sqrt{1-[\mathsf{s}_1^{(0)}(\omega)]^2}}. \quad \text{(A5b)}$$

From Eq. (A4) it follows that the spectra $S_x^{(0)}$ and $S_y^{(0)}$ are proportional to each other, and from Eqs. (A5) it follows that the correlation coefficient $\mu_{xy}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)$, evaluated for $\boldsymbol{\rho}_1=\boldsymbol{\rho}_2\equiv\boldsymbol{\rho}$, is independent of $\boldsymbol{\rho}$. Formulas (A4) and (A5) also show how the ratio $S_x^{(0)}(\boldsymbol{\rho},\omega)/S_y^{(0)}(\boldsymbol{\rho},\omega)$ of the spectra and the correlation coefficient $\mu_{xy}^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega)$ can be determined from the knowledge of the normalized Stokes parameters.

APPENDIX B: DERIVATION OF EQ. (3.3)

In this appendix we will evaluate the four-dimensional Fourier transform of the cross-spectral density matrix in the source plane, viz.,

$$\widetilde{W}_{ij}^{(0)}(\mathbf{f}_1,\mathbf{f}_2,\omega)=\frac{1}{(2\pi)^4}\int\int W_{ij}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)\exp[-i(\mathbf{f}_1\cdot\rho_1+\mathbf{f}_2\cdot\rho_2)]\mathrm{d}^2\rho_1\mathrm{d}^2\rho_2 \quad (i=x,y, \quad j=x,y), \quad \text{(B1)}$$

where

$$W_{ij}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\alpha_{ij}(\omega)S^{(0)}((\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)/2,\omega)\mu_{ij}^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega). \quad \text{(B2)}$$

Introducing the variables

$$\Delta\boldsymbol{\rho}=\boldsymbol{\rho}_2-\boldsymbol{\rho}_1, \quad \bar{\boldsymbol{\rho}}=(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)/2, \quad \text{(B3)}$$

we can express formula (B1) in the form

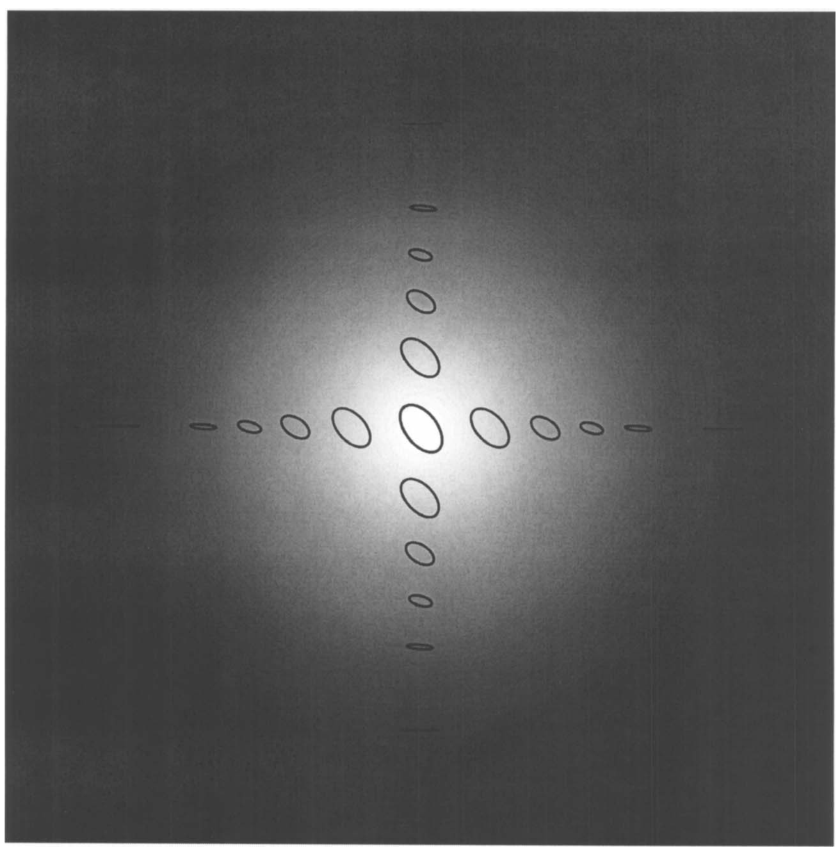


Fig. 6. Illustration of rotational symmetry of the spectral density and of polarization of the far field generated by the same source as in Fig. 5(a), plotted as functions of polar angle θ in the interval -3×10^{-3} rad $\leq\theta\leq 3\times10^{-3}$ rad. The black-to-white (gray-scale) background corresponds to the normalized values between 0 and 1 of the spectral density of the far field, normalized by its axial value. The ellipses superimposed on this background are the spectral polarization ellipses of the fully polarized portion of the far field. Their centers are chosen to be located along the two mutually orthogonal directions at points where the spectral density normalized by its axial value takes on the values 1, 0.85, 0.6, 0.4, 0.25, and 0.1.

$$\tilde{W}_{ij}^{(0)}(\mathbf{f}_1,\mathbf{f}_2,\omega)=\frac{\alpha_{ij}(\omega)}{(2\pi)^4}\int\int S^{(0)}(\bar{\boldsymbol{\rho}},\omega)\mu_{ij}^{(0)}(\Delta\boldsymbol{\rho},\omega)\exp[-i(\mathbf{f}_1\cdot(\bar{\boldsymbol{\rho}}-\Delta\boldsymbol{\rho}/2)+\mathbf{f}_2\cdot(\bar{\boldsymbol{\rho}}+\Delta\boldsymbol{\rho}/2))]\mathrm{d}^2\bar{\boldsymbol{\rho}}\mathrm{d}^2(\Delta\rho)$$

$$=\frac{\alpha_{ij}(\omega)}{(2\pi)^4}\int\int S^{(0)}(\bar{\boldsymbol{\rho}},\omega)\mu_{ij}^{(0)}(\Delta\boldsymbol{\rho},\omega)\exp[-i(\bar{\boldsymbol{\rho}}\cdot(\mathbf{f}_1+\mathbf{f}_2)+\Delta\boldsymbol{\rho}\cdot(\mathbf{f}_2-\mathbf{f}_1)/2)]\mathrm{d}^2\bar{\boldsymbol{\rho}}\mathrm{d}^2(\Delta\rho)$$

$$=\frac{\alpha_{ij}(\omega)}{(2\pi)^2}\int S^{(0)}(\bar{\boldsymbol{\rho}},\omega)\exp[-i\bar{\boldsymbol{\rho}}\cdot(\mathbf{f}_1+\mathbf{f}_2)]\mathrm{d}^2\bar{\boldsymbol{\rho}}\frac{1}{(2\pi)^2}\int\mu_{ij}^{(0)}(\Delta\boldsymbol{\rho},\omega)\exp[-i\Delta\boldsymbol{\rho}\cdot(\mathbf{f}_2-\mathbf{f}_1)/2]\mathrm{d}^2(\Delta\boldsymbol{\rho})$$

$$=\alpha_{ij}(\omega)\tilde{S}^{(0)}(\mathbf{f}_1+\mathbf{f}_2,\omega)\tilde{\mu}_{ij}^{(0)}((\mathbf{f}_2-\mathbf{f}_1)/2,\omega). \quad \text{(B4)}$$

This formula is analogous to the corresponding expression that pertains to the scalar case [see Eq. (5.3-19) in Ref. 1].

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12. We use the symbol $\mu_{ij}^{(0)}$ rather than $\eta_{ij}^{(0)}$ that was employed in Ref. 9 to avoid confusion with the scalar theory.
13. M. Born and E. Wolf, *Principles of Optics*, 7th expanded ed. (Cambridge U. Press, Cambridge, UK 1999).
14. The Stokes parameter $S_0^{(0)}(\boldsymbol{\rho},\omega)$ that we use for normalization is just the spectral density $S^{(0)}(\boldsymbol{\rho},\omega)$ of the source, given by Eq. (2.4).
15. A source that is linearly polarized along the y direction, say, obviously corresponds to the limit $\alpha(\omega)\to\infty$.
16. This result represents essentially a generalization to the far field generated by QH electromagnetic sources of the well-known van Cittert–Zernike theorem of the scalar theory.
17. The Stokes parameters of the far-zone field can also be calculated from the normalized ones, because the Stokes parameter used for normalization, is just the spectral density $S^{(\infty)}(\mathbf{r}s,\omega)$ of the far field, given by reciprocity relation (3.9).
18. The results derived in this section were found for some special cases in G. Piquero, R. Borghi, A. Mondello, and M. Santarsiero, "Far-field of beams generated by quasi-homogeneous sources passing through polarization gratings," Opt. Commun. **195**, 339–350 (2001).
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24. The parameter $\alpha(\omega)$ may be shown to be related to the angle $\psi(\omega)$ between the x axis and the direction of polarization at the source plane by the formula $\alpha(\omega)=\tan^{-1}|\psi(\omega)|$, $-\pi/2<\psi\leq\pi/2$. In practice, in order to control $\alpha(\omega)$, one can replace the beam splitter BS1 introduced in Ref. 19 with a polarizing beam splitter and place a polarization rotator (half-wave plate) in front of it. The parameter $\alpha(\omega)$ can then be determined by rotating the half-wave plate.
25. The parameters characterizing the spectral polarization ellipse at each point can also be calculated directly from the elements of the cross-spectral density matrix [see O. Korotkova and E. Wolf, "Polarization state changes in random electromagnetic beams on propagation," Opt. Commun. **246**, 35–43 (2005)].